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An extension of Milnor's $\bar{\mu}$-invariants

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Abstract

We study embeddings in a certain fixed, nontrivial homotopy class of one copy of the circle S^1 in any closed, aspherical, orientable, irreducible 3-dimensional Seifert fibered 3-manifold, M, and extract a collection of numerical concordance invariants from certain quotients of the fundamental group of the complement of the knot. (The property of the fundamental group of the target 3-manifold that is needed to produce these invariants is precisely the one that guarantees that the manifold is a Seifert fibered space, by the recently proved Seifert fibered space theorem.) These extensions of Milnor's $\bar{\mu}$-invariants detect "self-linking" phenomena that are nonsimply connected analogues to the "higher order" linking phenomena detected by the classical $\bar{\mu}$-invariants. In particular, they are obstructions to an embedding being concordant to a characteristic embedding. Examples of knots that realize some of these invariants are constructed for $M = T^3$, and a realization theorem is proved for any M, using a generic, ordinary Seifert fiber as the model of a "trivial" embedding.

Keywords: Knot; Link; Concordance; Cobordism; $\bar{\mu}$-invariants

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0. Introduction

In [8], Milnor developed a collection of isotopy invariants of links in $\mathbb{R}^3$. Work of Stallings [10] demonstrated that these $\bar{\mu}$-invariants are invariants of link concordance. $\bar{\mu}$-invariants and their interpretations and extensions have been the subject of important recent investigations [9,11,3]. We extend these to self-linking invariants of embeddings of one copy of the circle S^1 in any closed, aspherical 3-dimensional Seifert fibered space, M^3. The study of such embeddings was motivated by the examination of embeddings of

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S^1 in the 3-dimensional torus, $T^3 = S^1 \times S^1 \times S^1$, which was in turn motivated by the study of codimension-2 embeddings of the n-dimensional torus in the $(n+2)$-dimensional torus, $T^n \hookrightarrow T^{n+2}$, $n \geqslant 5$ in [6]. In spirit, these invariants represent a partial answer to a question raised by Milnor in [8]: how can the $\bar{\mu}$-invariants be generalized to arbitrary 3-manifolds? The property of $\pi_1 M^3$ needed to construct our invariants is the existence of an infinite cyclic normal subgroup. This is precisely the property that guarantees that M is a Seifert fibered space by the Seifert fibered space theorem (recently conjecture) proved independently and with different methods by Casson and Jungreis [2] and by Gabai [4]. As is the case with links, special phenomena arise for these embeddings in the classical dimension. Here these are nonsimply connected "self-linking" phenomena which are analogues of the higher order linking detected by Milnor's invariants. Like Milnor's $\bar{\mu}$-invariants, these are concordance invariants. Moreover, these invariants serve as obstructions to an embedding being characteristic.

In Section 1, we make necessary definitions and constructions. In Section 2, we derive our numerical invariants, and prove that certain inductively determined residue classes of these numbers are invariants of concordance. Finally, in Section 3, we construct explicit examples and prove a realization theorem. I would like to thank Sylvain Cappell for helpful discussions and suggestions.

1. Definitions and preliminaries

We study locally flat embeddings

$$f : S^1 \hookrightarrow M^3$$

where M^3 is a closed, aspherical, orientable, irreducible 3-dimensional Seifert fibered space such that the induced homomorphism

$$f_* : \mathbb{Z} \longrightarrow \pi_1 M^3$$

on fundamental groups is fixed and sends the generator of $\pi_1 S^1$ to a generator of the infinite cyclic normal subgroup of $\pi_1 M$. We study these embeddings under the equivalence relation of concordance.

Definition 1.1. Two embeddings f_0 and f_1 are said to be concordant if there is a submanifold V of $M^3 \times I$, homeomorphic to $S^1 \times I$, meeting the boundary transversely such that for $i = 0, 1$,

$$V \cap (M^3 \times i) = \operatorname{im}(f_i).$$

For any embedding $f : S^1 \hookrightarrow M^3$ satisfying the above condition, let E_f denote the complement of a tubular neighborhood $N_f \cong f(S^1) \times D^2$ of the embedding, and let $G_f = \pi_1(E_f)$. We have an epimorphism

$$h : G_f \twoheadrightarrow \pi_1 M$$

produced by filling in N_f. We describe $\pi_1 M$ by the following extension:

$$\mathbb{Z} \rightarrowtail \pi_1 M \overset{\psi}{\twoheadrightarrow} \Gamma.$$

Now the generator of the infinite cyclic normal subgroup can be represented by any based, ordinary Seifert fiber of M. Let

$$j_0 : S^1 \longrightarrow M$$

denote the embedding of our choice of an unbased Seifert fiber. This embedding will serve as our model of a "trivial" embedding of a circle in our Seifert fibered manifold, M. Let E_0 denote the complement of j_0. Let $G_0 = \pi_1 E_0$.

Definition 1.2. An embedding $f : S^1 \to M^3$ is termed j_0-characteristic or just characteristic if there is a map

$$\alpha : M^3 \longrightarrow M^3$$

such that

$$\alpha \circ f : S^1 \longrightarrow M^3$$

is identical to j_0, and $\alpha(M^3 - f(S^1)) \subset M^3 - S^1$.

The idea of a characteristic embedding also plays an important role in link theory. A link is characteristic if and only if it is a boundary link. Milnor's $\bar{\mu}$-invariants are obstructions to a link being concordant to a boundary link.

2. Numerical invariants of a knot in a Seifert fibered space

In order to study self-linking properties of embeddings $S^1 \hookrightarrow M^3$, we must examine the appropriate covering space. For any embedding f consider the epimorphism

$$\phi : G_f \twoheadrightarrow \Gamma$$

defined by $\phi = \psi \circ h$. Let $\overline{G}$ denote the kernel of ϕ, and let $\overline{E}$ denote the covering space of E_f corresponding to $\overline{G}$. Let $\overline{G}_0$, $\bar{\phi}_0$, and $\overline{E}_0$ represent the corresponding objects for G_0. $\overline{E}$ is the complement of the embedding

$$\bar{f} : \coprod_{g \in \Gamma} S^1_g \hookrightarrow S^1 \times \mathbb{R}^2$$

of disjoint circles indexed by the elements of Γ.

We need a presentation for $\overline{G}$. Let $\widetilde{G}$ denote the kernel of the "filling in" homomorphism h and let $\widetilde{E}$ denote the covering space of E_f corresponding to h. $\widetilde{E}$ is the complement of the embedding

$$\tilde{f} : \coprod_{g \in \Gamma} \mathbb{R}^1_g \hookrightarrow \mathbb{R}^3$$

of a disjoint union of real lines indexed by the elements of Γ in $\mathbb{R}^3$. $\widetilde{E}$ is an infinite cyclic cover of $\overline{E}$, so

$$\overline{G} = \widetilde{G} \rtimes \mathbb{Z}.$$

$\overline{G}$ is the semi-direct product of $\widetilde{G}$ and $\mathbb{Z} = \langle t \rangle$, with action given by conjugation. Let y_0 be a basepoint chosen for M^3 such that $y_0 \in \partial E_f$. Let $\bar{y}_0$ denote a lift of y_0 to the cover $S_1 \times \mathbb{R}^2$ of M^3. Let $\tilde{y}_0$ denote a lift of $\bar{y}_0$ to the universal cover $\mathbb{R}^3$ of M^3.

We wish to choose a based meridian, m, and a based longitude, l, of $\operatorname{im}(f)$ which are well defined up to simultaneous conjugation. In particular, there is a natural indeterminacy in choosing the longitude: if l_1 and l_2 are two based loops which are homotopic to $\operatorname{im}(f)$ in M^3, then $[l_1][l_2]^{-1} = [m]^k$. In the case of classical link theory, this indeterminacy is resolved by assuming that the longitude and meridian of each component of the link have linking number zero. Since the knots discussed here do not bound in M^3, we must pass to the appropriate covering space and construct the longitude there. We consider the embedding $\bar{f}$ above. The components of $\operatorname{im}(f)$ can be indexed by the elements of Γ in a manner consistent with our choice of the lift $\bar{y}_0$. Let N_e denote the tubular neighborhood of $[\operatorname{im}(\bar{f})]_e$ (the component of $\operatorname{im}(\bar{f})$ whose index is the identity element of Γ) such that N_e maps to N_f under the covering projection. We choose a meridian $m_e \subset \partial N_e$, i.e., a loop based at $\bar{y}_0$ via a path $p \subset \overline{E}$, such that m_e bounds a disc in $S^1 \times \mathbb{R}^2$ which intersects $[\operatorname{im}(\bar{f})]_e$ exactly once. We choose a longitude $l_e \subset \partial N_e$ to be a loop based via the same path $p \subset \overline{E}$ at $\bar{y}_0$ such that l_e is homotopic to $[\operatorname{im}(\bar{f})]_e$ in $S^1 \times \mathbb{R}^2$, and

$$\operatorname{pr}_e(h([l_e])) = 0$$

where

$$h : \pi_1(\overline{E}) \longrightarrow H_1(\overline{E}) = \mathbb{Z} \times \mathbb{Z}^{|\Gamma|}$$

is the abelianization homomorphism, and

$$\operatorname{pr}_e : \mathbb{Z} \times \mathbb{Z}^{|\Gamma|} \longrightarrow \mathbb{Z}$$

is projection onto the factor generated by $[m_e] \in H_1(\overline{E})$. Let $\pi : \overline{E} \to E_f$ be the covering projection. We define the meridian, m, and the longitude l of $\operatorname{im}(f)$ to be the images of m_e and l_e under π, i.e.,

$$m = \pi(m_e), \qquad l = \pi(l_e).$$

In the above extension of $\overline{G}$ as a semi-direct product, we choose our splitting so that $t = [l_e]$. It should be noted here that $\overline{G}_0 \cong \mathbb{Z} \times F_{|\Gamma|}$, the direct product of $\mathbb{Z}$ and a free group whose generators are indexed by the elements of Γ. For in this case, if $\langle t \rangle$ were not central in $\overline{G}_0$, it would not be normal in G_0, contradicting the fact that $\operatorname{im}(j_0)$ was chosen to be a Seifert fiber. As we shall see, this is our principal reason for choosing a Seifert fiber to be our "trivial" knot. All the invariants we shall define will be zero for this embedding.

Let D be a fundamental region of the universal cover of M^3 such that $\tilde{y}_0 \in \partial D$. If necessary, perturb f so that $\tilde{f}$ intersects ∂D transversly. We then choose a regular projection of $D \cap \operatorname{im}(\tilde{f})$ onto a distant plane in $\mathbb{R}^3$ and write a Wirtinger presentation for the fundamental group of $D - (D \cap N_{\tilde{f}})$, using $\tilde{y}_0$ as the basepoint. This is a finitely presented group,

$$\langle m_e^{ab} \mid R_e^{ab} \rangle,$$

where e denotes the identity element of Γ, $a = 1, \ldots, n$, $b = 1, \ldots, r_a$, and n is the number of components of $D \cap \operatorname{im}(\tilde{f})$, r_a the number of components in the projection of the component a. As in the Wirtinger presentation for links, the m_e^{ab} represent meridians for the connected components of the projection and the relations account for the crossings in the projection. We obtain a presentation of $\tilde{G}$ by replicating this presentation for each fundamental domain D_g, $g \in \Gamma$, identifying generators as necessary. Thus,

$$\tilde{G} = \langle \{m_g^{ab} : g \in \Gamma\} \mid \{R_g^{ab} : g \in \Gamma\},\ m_h^r = m_k^s \rangle.$$

Here, h and k are elements of Γ such that D_h and D_k are adjacent fundamental domains, and m_h^r, m_k^s represent components of $\operatorname{im}(\tilde{f})$ connected over $\partial D_h \cap \partial D_k$. Using this presentation of $\tilde{G}$ we are prepared to state the following theorem.

Theorem 2.1. *The group $\overline{G}/\overline{G}_q$ has the presentation*

$$\langle \{x_g : g \in \Gamma\},\ t \mid \{t^{-1} x_g t = w_g^{-1} x_g w_g : g \in \Gamma\},\ F_q \rangle$$

where $t = [l_e]$, the w_g are words in the generators $\{x_h : h \in \Gamma\}$ and F_q is the qth lower central subgroup of the (possibly infinitely generated) free group $F_{|\Gamma|} = \langle \{x_h : h \in \Gamma\} \rangle$, whose generators are indexed by the elements of Γ.

Proof. We have the extension

$$\tilde{G} \rightarrowtail \overline{G} \twoheadrightarrow \mathbb{Z}$$

given by filling in the components of $\operatorname{im}(\tilde{f})$. Thus,

$$\overline{G} = \tilde{G} \rtimes \mathbb{Z} = \langle \{m_{ab}^g : g \in \Gamma\},\ t \mid \{R_{ab}^g\},\ \{m_r^h = m_s^k\},\ \{t^{-1} m_{ab}^g t = \phi(m_{ab}^g)\} \rangle$$

where

$$\phi : \tilde{G} \to \tilde{G}.$$

Because the generators, m_{ab}^g, of $\tilde{G}$ are meridians of $\operatorname{im}(\tilde{f})$, and the action of t is by translation,

$$\phi(m_{ab}^g) = (m_{ab}^g)^{w_{ab}^g} = (w_{ab}^g)^{-1} m_{ab}^g w_{ab}^g,$$

i.e., the image of each meridian under ϕ is a conjugate of that meridian by another element of $\tilde{G}$.

Now to compute the quotient $\overline{G}/\overline{G}_q$, we need only consider the effect of the quotient on the factor $\widetilde{G}$. Since, $\widetilde{G} = \pi_1(E_{\tilde{f}})$, we have a map

$$\sigma : F_{|\Gamma|} \longrightarrow \widetilde{G}$$

from the free group to $\widetilde{G}$ given by choosing meridians for each component of $\mathrm{im}(\tilde{f})$. A Mayer–Vietoris argument shows that $H_2(E_{\tilde{f}}) = 0$, so by Hopf's theorem, $H_2(\widetilde{G}) = 0$, and Stallings' theorem can be applied to tell us that

$$\widetilde{G}/\widetilde{G}_q = F_{|\Gamma|}/F_q$$

and the theorem follows. $\square$

We next consider the Magnus expansion, (see [5]), of the words w_g, obtained by defining a mapping

$$\mathcal{M} : F_{|\Gamma|} \longrightarrow \mathbb{P}$$

where $\mathbb{P}$ is the group of units of the ring of formal power series in countably many noncommuting indeterminates, $\{\kappa_g : g \in \Gamma\}$, and $\mathcal{M}$ is specified by stipulating that

$$\mathcal{M}(x_g) = 1 + \kappa_g,$$

$$\mathcal{M}(x_g^{-1}) = 1 - \kappa_g + \kappa_g^2 - \kappa_g^3 + \cdots$$

and we extend to a homomorphism by multiplying polynomials. We write the expansion

$$\mathcal{M}(w_g) = 1 + \sum \tau(h_1, \ldots, h_s, g)\kappa_{h_1} \cdots \kappa_{h_s}$$

where $h_1, \ldots, h_s$ are elements of Γ. It is clear that all the coefficients $\tau(h_1, \ldots, h_s, g)$ are zero for our trivial embedding, j_0.

Let $\Delta(h_1, \ldots, h_r)$ denote the greatest common divisor of $\tau(k_1, \ldots, k_s)$, where $k_1, \ldots, k_s$, $2 \leqslant s < r$ is to range over all sequences of elements of Γ obtained by cancelling at least one of $h_1, \ldots, h_r$, and permuting the remaining elements cyclicly. Let $\bar{\tau}(h_1, \ldots, h_r)$ denote the residue class of $\tau(h_1, \ldots, h_r)$ modulo $\Delta(h_1, \ldots, h_r)$.

Theorem 2.2. *For any embedding*

$$f : S^1 \longrightarrow M^3$$

(i) *The residue classes $\bar{\tau}(h_1, \ldots, h_r)$ are concordance invariants of $\bar{f}$, provided that $r \leqslant q$.*

(ii) *The residue classes $\bar{\tau}(h_1, \ldots, h_r)$ are concordance invariants of f, $r \leqslant q$.*

(iii) *If f is a characteristic embedding, then all the numbers $\bar{\tau}(h_1, \ldots, h_r)$, are zero.*

Proof. (i) The classes x_g and t are well defined up to conjugacy. We must prove that the residue class $\bar{\tau}(h_1, \ldots, h_s, g)$, $s < q$ is unaltered if

(1) x_g is replaced by a conjugate.

(2) w_g is replaced by a conjugate.

(3) w_g is multiplied by a relation of the form $[tw_h, x_h]$.

(4) w_g is multiplied by an element of F_q.

In the Magnus power series in countably many indeterminates, let I_g denote the set of all elements

$$\sum \nu(k_1,\ldots,k_s)\kappa_{k_1},\ldots,\kappa_{k_s}$$

with coefficients satisfying

$$\nu(k_1,\ldots,k_s) \equiv 0 \mod \Delta(k_1,\ldots,k_s,g)$$

for all $k_1,\ldots,k_s$, with $s<q$, and no restrictions when $s \geqslant q$.

I_g is a 2-sided ideal, because if $\nu(k_1,\ldots,k_s)\kappa_{k_1},\ldots,\kappa_{k_s}$ is a monomial in I_g and $\lambda\kappa_{j_1},\ldots,\kappa_{j_r}$ is an arbitrary element, then either $s \geqslant q$ and the product obtained by right multiplication by $\lambda\kappa_{j_1},\ldots,\kappa_{j_r}$ is trivially in I_g or

$$\nu(k_1,\ldots,k_s) \equiv 0 \mod \Delta(k_1,\ldots,k_s,g)$$

and the product

$$\lambda\nu(k_1,\ldots,k_s)\kappa_{k_1},\ldots,\kappa_{k_s}\kappa_{j_1},\ldots,\kappa_{j_r}$$

is clearly in I_g as

$$\nu(k_1,\ldots,k_s) \equiv 0 \mod \Delta(k_1,\ldots,k_s,j_1,\ldots,j_r,g).$$

A similar argument works for left multiplication.

We will use the following fact about I_g: if w_g and v_g are two words in $\{x_h : h \in \Gamma\}$, and $w_g \equiv v_g \mod I_g$, then w_g and v_g determine the same residue classes $\bar{\tau}(k_1,\ldots,k_s,g)$, $s<q$.

The proofs of (1)-(4) will use the following facts.

(a) Let $1+W_g$ denote the Magnus expansion of w_g. Then

$$W_g\kappa_h \equiv \kappa_h W_g \equiv 0 \mod I_g$$

for any κ_h. This follows from the congruences

$$\tau(k_1,\ldots,k_s,g) \equiv 0 \mod \Delta(k_1,\ldots,k_s,g,h),$$

$$\tau(k_1,\ldots,k_s,g) \equiv 0 \mod \Delta(h,k_1,\ldots,k_s,g).$$

(b) If one or more factors κ_h are inserted in the term

$$\tau(k_1,\ldots,k_s,g)\kappa_{k_1},\ldots,\kappa_{k_s}$$

then the resulting term is congruent to zero mod I_g. This follows from the congruences

$$\tau(k_1,\ldots,k_s,g) \equiv 0 \mod \Delta(k_1,\ldots,k_c,h,k_{c+1},\ldots,k_s).$$

(c) Let $1+W_h$ denote the Magnus expansion of w_h. Then

$$\kappa_h W_h \equiv W_h \kappa_h \equiv 0 \mod I_g.$$

This follows from the congruences

$$\tau(k_1,\ldots,k_s,g) \equiv 0 \mod \Delta(k_1,\ldots,k_s,h,g),$$

$$\tau(k_1,\ldots,k_s,h) \equiv 0 \mod \Delta(h,k_1,\ldots,k_s,g).$$

(d) $\mathcal{M}(F_q) \equiv 0 \mod I_g$. This follows from Magnus' work on the Magnus expansion [5], namely that $g \in F_q$ if and only if it's Magnus expansion has only terms of degree $\geqslant q$.

Proof of (1): Suppose that x_h is replaced by $x_h' = x_k x_h x_k^{-1}$. Then in the expression $\mathcal{M}(x_h') = 1 + \kappa_h'$,

$$\begin{aligned}\kappa_h' &= (1+\kappa_k)(\kappa_h)(1-\kappa_k+\kappa_k^2-\cdots)\\ &= \kappa_h + \text{terms involving } \kappa_k\kappa_h \text{ or } \kappa_h\kappa_k.\end{aligned}$$

In the Magnus expansion of w_g in terms of κ_j, $j \neq h$, and κ_h', it follows that the second collection of terms in the expansion of κ_h' gives rise to terms which are zero mod I_g, by (b). Hence the residue classes are unaltered. Any conjugation is a sequence of the above.

Proof of (2): Suppose that w_g is replaced by $x_h w_g x_h^{-1}$. Then the polynomial W_g is replaced by

$$\begin{aligned}(1+\kappa_h)(W_g)(1-\kappa_h+\kappa_h^2-\cdots) &= W_g + \text{terms involving } \kappa_h W_g \text{ or } W_g\kappa_h\\ &\equiv W_g \mod I_g, \quad \text{by (a)}.\end{aligned}$$

Any conjugation is a sequence of the above.

Proof of (3): The relations $t^{-1}x_h t = w_h^{-1}x_h w_h$ in $\overline{G}/\overline{G}_q$ can be written in the form $[tw_h^{-1}, x_h]$. If w_g is multiplied by such a relation, to obtain $w_g' = [tw_h^{-1}, x_h]w_g$, we must rewrite the relation in order to determine the magnus expansion of w_g'. Assume $w_h = x_{k_1}^{\alpha_1}\ldots x_{k_p}^{\alpha_p}$ (repeats in the indices are possible). Then

$$\begin{aligned}[tw_h^{-1}, x_h] &= tw_h^{-1}x_h w_h t^{-1} x_h^{-1}\\ &= t(x_{k_p}^{-\alpha_p}\ldots x_{k_1}^{-\alpha_1})x_h(x_{k_1}^{\alpha_1}\ldots x_{k_p}^{\alpha_p})t^{-1}x_h^{-1}\\ &= ((tx_{k_p}^{-\alpha_p}t^{-1})(tx_{k_{p-1}}^{-\alpha_{p-1}}t^{-1})\ldots(tx_{k_1}^{-\alpha_1}t^{-1}))\\ &\quad (tx_h t^{-1})((tx_{k_1}^{\alpha_1}t^{-1})(tx_{k_2}^{\alpha_2}t^{-1})\ldots(tx_{k_p}^{\alpha_p}t^{-1}))x_h^{-1}\\ &= ((w_{k_p}x_{k_p}^{-\alpha_p}w_{k_p})(w_{k_{p-1}}x_{k_{p-1}}^{-\alpha_{p-1}}w_{k_{p-1}})\ldots(w_{k_1}x_{k_1}^{-\alpha_1}w_{k_1}))\\ &\quad (w_h x_h w_h^{-1})((w_{k_1}x_{k_1}^{\alpha_1}w_{k_1})(w_{k_2}x_{k_2}^{\alpha_2}w_{k_2})\ldots(w_{k_p}x_{k_p}^{\alpha_p}w_{k_p}))x_h^{-1}\\ &= [((w_{k_p}x_{k_p}^{-\alpha_p}w_{k_p})(w_{k_{p-1}}x_{k_{p-1}}^{-\alpha_{p-1}}w_{k_{p-1}})\ldots(w_{k_1}x_{k_1}^{-\alpha_1}w_{k_1}))w_h, x_h]\\ &= [w_h^*, x_h].\end{aligned}$$

Consider the identity:

$$\begin{aligned}\mathcal{M}(x_h)\mathcal{M}(w_h^*) - \mathcal{M}(w_h^*)\mathcal{M}(x_h) &= (1+\kappa_h)(1+W_h^*) - (1+W_h^*)(1+\kappa_h)\\ &= \kappa_h W_h^* - W_h^*\kappa_h.\end{aligned}$$

Then (3) follows from the congruence

$$\begin{aligned}\mathcal{M}([w_h^*, x_h]) &= 1 - \mathcal{M}(x_h w_h^* (w_h^*)^{-1} x_h^{-1}) + \mathcal{M}([w_h^*, x_h]) \\ &= 1 - (\kappa_h W_h^* - W_h^* \kappa_h)(\mathcal{M}((w_h^*)^{-1} x_h^{-1})) \\ &\equiv 1 \mod I_g, \quad \text{by (c)}.\end{aligned}$$

Right multiplication by a relation of the form $[tw_h^{-1}, x_h]$ is handled similarly.

Proof of (4): This follows from (d). We have thus proved (i).

(ii) (It is proved by Casson, [1] that quotients of the fundamental group of a cover of a link in any 3-manifold by terms of its lower central series are concordance invariants of the link. Here we provide a proof using Stallings' theorem.) Let $f_1, f_2 : S^1 \to M^3$ be concordant embeddings, and let

$$F : S^1 \times I \longrightarrow M^3 \times I$$

be a concordance between them. Let $\Lambda \stackrel{\text{def}}{=} F(S^1 \times I) \subset M^3 \times I$. Let $\overline{E}_i = S^1 \times \mathbb{R}^2 - \mathrm{im}(\bar{f}_i)$, $i = 1, 2$, and $\Delta = S^1 \times \mathbb{R}^2 \times I - \overline{\Lambda}$. We have a commutative diagram:

$$\begin{array}{ccc} H^{3-r}(\mathrm{im}(\bar{f}_i)) & \longrightarrow & H^{4-r}(\overline{\Lambda}, \partial\overline{\Lambda}) \\ \downarrow & & \downarrow \\ H_r(S^1 \times \mathbb{R}^2, \overline{E}_i) & \longrightarrow & H_r(S^1 \times \mathbb{R}^2 \times I, \Delta) \end{array}$$

in which the vertical maps are Alexander duality isomorphisms. The top coboundary map is an isomorphism, so the bottom map is also an isomorphism. It follows that inclusion induces isomorphisms

$$H_r(\overline{E}_i) \to H_r(\Delta).$$

Thus Stallings' theorem can be applied to show that

$$\overline{G^1}/\overline{G^1_q} \cong \overline{G^2}/\overline{G^2_q}.$$

These isomorphisms preserve our peripheral structures: lifts of the meridian and longitude of $\mathrm{im}(f)$. Let $\mathcal{N}_g = S^1 \times I \times D^2$ be a tubular neighborhood of $[\mathrm{im}(\overline{F})]_g \cong S^1 \times I$. Let $N_g^1 = \mathcal{N}_g \cap (M \times 0)$ and $N_g^2 = \mathcal{N}_g \cap (M \times 1)$. Let $m_g^1, l_g^1 \subset N_g^1$ be lifts of a choice of meridian and longitude of $[\mathrm{im}(\bar{f}_1)]_g$. We then homotope m_g^1 and l_g^1 in $\mathcal{N}_g$ to loops m_g^2, l_g^2 in N_g^2. These loops are a meridian-longitude pair for $[\mathrm{im}(\bar{f}_2)]_g$. Since the isomorphisms of Stallings' theorem are induced by the inclusions $\overline{E}_i \subset S^1 \times \mathbb{R}^2$, these isomorphisms preserve our peripheral structures, and the words w_g are concordance invariants modulo the indeterminacies (1)–(4) discussed in the proof of (i). Thus the residue classes $\bar{\tau}(h_1, \ldots, h_r)$ are concordance invariants of f if $r < q$.

(iii) If f is characteristic (see Definition 1.2), then the restriction of the map α to E_f,

$$\hat{\alpha} : E_f \longrightarrow E_0$$

takes E_f to the complement of the inclusion, j_0, of the Seifert fiber. We then have an induced map

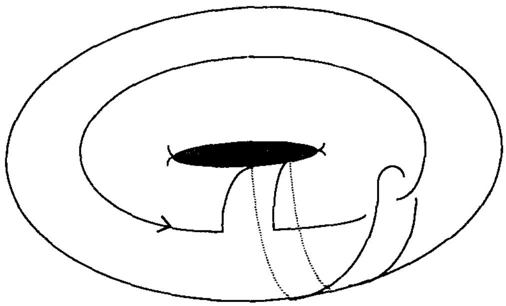

Fig. 1.

$$(\hat{\alpha})_* : G_f \longrightarrow G_0$$

and we have the diagram:

$$\begin{array}{ccc} \overline{G} & \xrightarrow{\theta} & \mathbb{Z} \times F_{|\Gamma|} \\ \downarrow & & \downarrow \\ G & \longrightarrow & G_0 \\ \phi \downarrow & & \downarrow \phi_0 \\ \Gamma & = & \Gamma \end{array}$$

The existence of the homomorphism θ demonstrates that all the words $w_g = e$, and all the numbers $\tau(k_1, \ldots, k_s, g) = 0$. □

3. Discussion, examples, and a realization theorem

In order to study some concrete examples, we consider embeddings $f : S^1 \longrightarrow T^3$. In this case, $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ and it acts on the components of $\mathrm{im}(\bar{f})$ by translation. Figs. 1 and 2 are pictures that illustrate examples of embeddings with nonzero $\bar{\tau}$ invariants. These should be thought of as pictures of a thin "slice", homeomorphic to $T^2 \times (e^{-ic}, e^{ic})$, of the 3-dimensional torus T^3.

The embedding in Fig. 1, let us term it f_1, exhibits a simple self-linking phenomenon. Our goal is to use the $\bar{\tau}$-invariants to detect the analogue of the linking number.

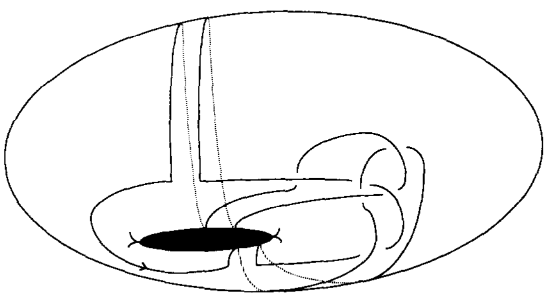

Fig. 2.

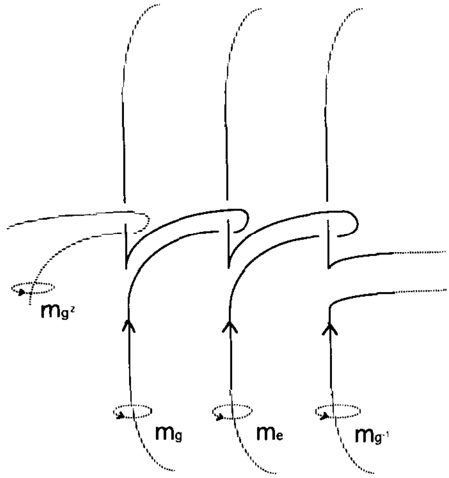

Fig. 3.

The embedding in Fig. 2, f_2, exhibits a higher order self-linking phenomenon. Indeed, it is apparent that f_2 "looks like" the borromean rings. Our goal is to use the $\bar{\tau}$-invariants to demonstrate that f_2 is not concordant to j_0, or even to any characteristic embedding and to find out how "deep" in the fundamental group of its complement the obstruction lies.

Recall that our invariants reside in quotients of the subgroup of the fundamental group corresponding to the natural homomorphism $\phi : G_f \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}$ or, alternatively, the fundamental group of the complement of $\bar{f}$. It is illustrative to consider Figs. 3 and 4. These are pictures of a few components of $\operatorname{im}(\bar{f}_i)$, $i = 1, 2$, in the open cylinder $S^1 \times \mathbb{R}^2$, with meridians subscripted by the elements of $\mathbb{Z} \oplus \mathbb{Z}$ which index their respective components. In both these pictures, we may take $g = (1, 0) \in \mathbb{Z} \oplus \mathbb{Z}$. The portions of the components which are not drawn loop around the S^1 factor. Such pictures motivate the term "self-linking".

Indeed, it is these covers which allow for the easy computation of the $\bar{\tau}$-invariants. This is accomplished, in essence, by writing down the relations between the longitudes. Let l_e, l_g, etc., denote longitudes of the components shown in Figs. 3 and 4. Since m_g and l_g lie on the same torus, it is clear that $[l_g, m_g] = e_{\bar{G}}$.

Examining Fig. 3 allows us to see the relation between l_g and l_e, namely that $l_g = l_e m_{g^{-1}} m_{g^2}^{-1}$. Thus we have,

$$[l_g, m_g] = [l_e m_{g^{-1}} m_{g^2}^{-1}, m_g] = e_{\bar{G}},$$

$$l_e m_{g^{-1}} m_{g^2}^{-1} m_g m_{g^2} m_{g^{-1}}^{-1} l_e^{-1} m_g^{-1} = e_{\bar{G}},$$

$$l_e m_{g^{-1}} m_{g^2}^{-1} m_g m_{g^2} m_{g^{-1}}^{-1} l_e^{-1} = m_g,$$

and finally,

$$l_e^{-1} m_g l_e = (m_{g^{-1}} m_{g^2}^{-1}) m_g (m_{g^2} m_{g^{-1}}^{-1}).$$

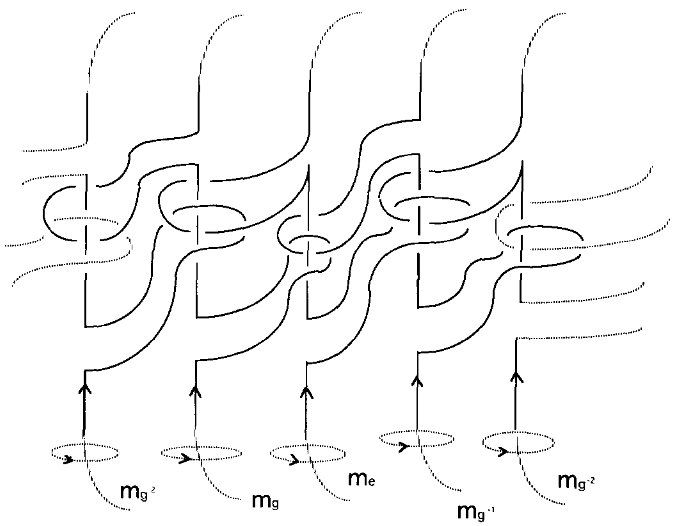

Fig. 4.

It is clear that in our notation, $w_g = m_{g^2} m_{g^{-1}}^{-1}$. We consider the Magnus expansion

$$\begin{aligned}\mathcal{M}(w_g) &= \mathcal{M}(m_{g^2} m_{g^{-1}}^{-1}) \\ &= \mathcal{M}(m_{g^2})\mathcal{M}(m_{g^{-1}}^{-1}) \\ &= (1+\kappa_{g^2})(1-\kappa_{g^{-1}}+\kappa_{g^{-1}}^2-\cdots) \\ &= 1+\kappa_{g^2}-\kappa_{g^{-1}}-\kappa_{g^2}\kappa_{g^{-1}}+\kappa_{g^{-1}}^2+\cdots.\end{aligned}$$

We conclude that the only nonvanishing $\bar{\tau}$-invariants of f_1 with respect to the component indexed by g are as follows:

$$\bar{\tau}(g^{-1}, g) = -1,$$

$$\bar{\tau}(g^2, g) = 1.$$

Similar relations between longitudes allow computation of $\bar{\tau}$-invariants of f_1 with respect to other elements of $\mathbb{Z} \oplus \mathbb{Z}$.

A similar, although albeit somewhat more elaborate, computation allows us to compute the $\bar{\tau}$-invariants for f_2. We note again that $[l_g, m_g] = e_{\bar{G}}$.

By studying Fig. 4, we observe that

$$l_g = l_e[m_{g^{-2}}, m_{g^{-1}}^{-1}][m_{g^{-1}}^{-1}, m_e^{-1}][m_{g^2}, m_g^{-1}].$$

We then have

$$[l_g, m_g] = [l_e[m_{g^{-2}}, m_{g^{-1}}^{-1}][m_{g^{-1}}^{-1}, m_e^{-1}][m_{g^2}, m_g^{-1}], m_g] = e_{\bar{G}},$$

$$\begin{aligned}&l_e[m_{g^{-2}}, m_{g^{-1}}^{-1}][m_{g^{-1}}^{-1}, m_e^{-1}][m_{g^2}, m_g^{-1}] \\ &\quad m_g[m_g^{-1}, m_{g^2}][m_e^{-1}, m_{g^{-1}}^{-1}][m_{g^{-1}}^{-1}, m_{g^{-2}}]l_e^{-1} = m_g,\end{aligned}$$

and

$$\begin{aligned} l_e^{-1} m_g l_e = {} & ([m_{g^{-2}}, m_{g^{-1}}^{-1}][m_{g^{-1}}^{-1}, m_e^{-1}][m_{g^2}, m_g^{-1}]) \\ & m_g([m_g^{-1}, m_{g^2}][m_e^{-1}, m_{g^{-1}}^{-1}][m_{g^{-1}}^{-1}, m_{g^{-2}}]). \end{aligned}$$

Thus we see that $w_g = [m_g^{-1}, m_{g^2}][m_e^{-1}, m_{g^{-1}}^{-1}][m_{g^{-1}}^{-1}, m_{g^{-2}}]$, and we consider its Magnus expansion:

$$\begin{aligned} \mathcal{M}(w_g) &= \mathcal{M}([m_g^{-1}, m_{g^2}][m_e^{-1}, m_{g^{-1}}^{-1}][m_{g^{-1}}^{-1}, m_{g^{-2}}]) \\ &= \mathcal{M}([m_g^{-1}, m_{g^2}])\mathcal{M}([m_e^{-1}, m_{g^{-1}}^{-1}])\mathcal{M}([m_{g^{-1}}^{-1}, m_{g^{-2}}]) \\ &= (1 - \kappa_g + \kappa_g^2 - \cdots)(1 + \kappa_{g^2})(1 + \kappa_g)(1 - \kappa_{g^2} + \kappa_{g^2}^2 - \cdots)\cdots \\ &= (1 + \kappa_g\kappa_{g^2} - \kappa_{g^2}\kappa_g - \cdots)(1 + \kappa_e\kappa_{g^{-1}} - \kappa_{g^{-1}}\kappa_e - \cdots) \\ &\quad\ (1 + \kappa_{g^{-1}}\kappa_{g^{-2}} - \kappa_{g^{-2}}\kappa_{g^{-1}} - \cdots) \\ &= 1 + \kappa_g\kappa_{g^2} - \kappa_{g^2}\kappa_g + \kappa_e\kappa_{g^{-1}} - \kappa_{g^{-1}}\kappa_e + \kappa_{g^{-1}}\kappa_{g^{-2}} - \kappa_{g^{-2}}\kappa_{g^{-1}} \cdots. \end{aligned}$$

We conclude that the only nonvanishing $\bar{\tau}$-invariants of f_2 with respect to the component indexed by g are as follows:

$$\begin{aligned} &\bar{\tau}(g, g^2, g) = 1, && \bar{\tau}(g^2, g, g) = -1, \\ &\bar{\tau}(e, g^{-1}, g) = 1, && \bar{\tau}(g^{-1}, e, g) = -1, \\ &\bar{\tau}(g^{-1}, g^{-2}, g) = 1, && \bar{\tau}(g^{-2}, g^{-1}, g) = -1. \end{aligned}$$

Similar relations between longitudes allow computation of $\bar{\tau}$-invariants of f_2 with respect to other elements of $\mathbb{Z} \oplus \mathbb{Z}$.

We will now prove a realization theorem in the general setting of knots in a Seifert fibered 3-manifold, $f : S^1 \to M^3$.

Theorem 3.1. *Given any word $g \in F_q \lhd F_{|\Gamma|}$ in the qth lower central series of $F_{|\Gamma|} = \langle\{x_g : g \in \Gamma\}\rangle$ such that*

$$g = [x_{h_1}^{k_1}, [x_{h_2}^{k_2}, [\ldots, [x_{h_{q-1}}^{k_{q-1}}, x_{h_q}^{k_q}]\ldots], \quad h_i \neq h_j,$$

there is an embedding $f : S^1 \longrightarrow M^3$ with

$$\bar{\tau}(h_1, \ldots, h_q, h) = \prod_{i=1}^{q} k_i, \quad \textit{for any } h \neq h_i, i = 1, \ldots, q$$

and all $\bar{\tau}$-invariants of length less than $q+1$ are zero. (Certain other $\bar{\tau}$-invariants of length equal to $q+1$ are also nonzero.)

Proof. Recall that for the fundamental group of the complement of the inclusion, j_0, of the Seifert fiber of M,

$$\mathbb{Z} \times F_{|\Gamma|} = \overline{G}_0 \lhd G_0 = \pi_1 E_0.$$

We choose a geometric representative for $g \in F_{|\Gamma|}$ as an element of $\pi_1 E_0$. Denote it by α. We take the band connect sum of $\mathrm{im}(j_0)$ and α along a band that intersects ∂N_{j_0} exactly once.

$\alpha\#\operatorname{im}(j_0)$ defines an embedding $f: S^1 \to M$. It is evident that in $\overline{E}$,

$$l_h = l_e g'(\text{words in } \overline{G}_q \text{ not including } m_{h_1}, \ldots, m_{h_q})$$

where

$$g' = [m_{h_1}^{k_1}, [m_{h_2}^{k_2}, [\ldots, [m_{h_{q-1}}^{k_{q-1}}, m_{h_q}^{k_q}] \ldots]$$

is a word in the meridians of $\bar{f}$. Using the computational methods above, it is evident that

$$\bar{\tau}(h_1, \ldots, h_q, h) = \prod_{i=1}^{q} k_i.$$

From Magnus' work on the Magnus expansion [5, Theorem 5.7], the fact that $w_h \in \overline{G}_q$ implies that all $\bar{\tau}$-invariants of length less than $q+1$ are zero. □

4. Concluding remarks

In a subsequent paper, we will develop an obstruction theoretic approach to these invariants, and prove further realization results.

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